

# A $(p, q)$ -ANALOGUE OF POLY-EULER POLYNOMIALS AND SOME RELATED POLYNOMIALS

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**ABSTRACT.** In the present article, we introduce a  $(p, q)$ -analogue of the poly-Euler polynomials and numbers by using the  $(p, q)$ -polylogarithm function. These new sequences are generalizations of the poly-Euler numbers and polynomials. We give several combinatorial identities and properties of these new polynomials. Moreover, we show some relations with the  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Cauchy polynomials. The  $(p, q)$ -analogues generalize the well-known concept of the  $q$ -analogue.

## 1. INTRODUCTION

The Euler numbers are defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The sequence  $(E_n)_n$  counts the numbers of alternating  $n$ -permutations. A  $n$ -permutation  $\sigma$  is alternating if the  $n-1$  differences  $\sigma(i+1) - \sigma(i)$  for  $i = 1, 2, \dots, n-1$  have alternating signs. For example, (1324) and (3241) are alternating permutations (cf. [9]).

The Euler polynomials are given by the generating function

$$(1) \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Note that  $E_n = 2^n E_n(1/2)$ .

Many kinds of generalizations of these numbers and polynomials have been presented in the literature (see, e.g., [32]). In particular, we are interested in the poly-Euler numbers and polynomials (cf. [11, 14, 15, 27]).

The poly-Euler polynomials  $E_n^{(k)}(x)$  are defined by the following generating function

$$\frac{2\text{Li}_k(1 - e^{-t})}{1 + e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}),$$

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where

$$(2) \quad \text{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

is the  $k$ -th polylogarithm function. Note that if  $k = 1$ , then  $\text{Li}_1(t) = -\log(1-t)$ , therefore  $E_n^{(1)}(x) = E_{n-1}(x)$  for  $n \geq 1$ .

It is also possible to define the poly-Bernoulli and poly-Cauchy numbers and polynomials from the  $k$ -th polylogarithm function. In particular, the poly-Bernoulli numbers  $B_n^{(k)}$  were introduced by Kaneko [16] by using the following generating function

$$(3) \quad \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

If  $k = 1$  we get  $B_n^{(1)} = (-1)^n B_n$  for  $n \geq 0$ , where  $B_n$  are the Bernoulli numbers. Remember that the Bernoulli numbers  $B_n$  are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The poly-Bernoulli numbers and polynomials have been studied in several papers; among other references, see [2, 3, 6, 7, 20, 21].

The poly-Cauchy numbers of the first kind  $c_n^{(k)}$  were introduced by the first author in [18]. They are defined as follows

$$(4) \quad c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (t_1 \cdots t_k)_n dt_1 \cdots dt_k$$

where  $(x)_n = x(x-1) \cdots (x-n+1)$  ( $n \geq 1$ ) with  $(x)_0 = 1$ . Moreover, its exponential generating function is

$$(5) \quad \text{Lif}_k(\ln(1+t)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

where

$$\text{Lif}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}$$

is the  $k$ -th polylogarithm factorial function. For more properties about these numbers see for example [7, 19, 20, 21, 22, 23]. If  $k = 1$ , we recover the Cauchy numbers  $c_n^{(1)} = c_n$ . The Cauchy numbers  $c_n$  were introduced in [9] by the generating function

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

A generalization of the above sequences was done recently in [20], using the  $k$ -th  $q$ -polylogarithm function and the Jackson's integral. In particular, the  $q$ -poly-Bernoulli numbers are defined by

$$(6) \quad \frac{\text{Li}_{k,q}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)} \frac{t^n}{n!}, \quad (k \in \mathbb{Z}, n \geq 0, 0 \leq q < 1),$$

where

$$\text{Li}_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}$$

is the  $k$ -th  $q$ -polylogarithm function (cf. [25]), and  $[n]_q = \frac{1-q^n}{1-q}$  is the  $q$ -integer (cf. [32]). Note that  $\lim_{q \rightarrow 1} [x]_q = x$ ,  $\lim_{q \rightarrow 1} B_{n,q}^{(k)} = B_n^{(k)}$  and  $\lim_{q \rightarrow 1} \text{Li}_{k,q}(x) = \text{Li}_k(x)$ .

The  $q$ -poly-Cauchy numbers of the first kind  $c_{n,q}^{(k)}$  are defined by using the Jackson's  $q$ -integral (cf. [1])

$$(7) \quad c_{n,q}^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (t_1 \cdots t_k)_n d_q t_1 \cdots d_q t_k$$

where

$$\int_0^x f(t) d_q t = (1 - q)x \sum_{n=0}^{\infty} f(q^n x) q^n.$$

Moreover, its exponential generating function is

$$\text{Lif}_{k,q}(\ln(1 + t)) = \sum_{n=0}^{\infty} c_{n,q}^{(k)} \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

where

$$(8) \quad \text{Lif}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! [n+1]_q^k}$$

is the  $k$ -th  $q$ -polylogarithm factorial function (cf. [20, 17]). Note that  $\lim_{q \rightarrow 1} c_{n,q}^{(k)} = c_n^{(k)}$  and  $\lim_{q \rightarrow 1} \text{Lif}_{k,q}(t) = \text{Lif}_k(t)$ .

In this paper, we introduce a  $(p, q)$ -analogue of the poly-Euler polynomials by

$$(9) \quad \frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} e^{xt} = \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z})$$

with  $p$  and  $q$  real numbers such that  $0 < q < p \leq 1$ , and

$$\text{Li}_{k,p,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_{p,q}^k}$$

is an extension of the  $q$ -polylogarithm function and we call it the  $(p, q)$ -polylogarithm function. The polynomials  $E_{n,p,q}^{(k)}(0) := E_{n,p,q}^{(k)}$  are called  $(p, q)$ -poly-Euler numbers. The polynomial  $[n]_{p,q} = \frac{p^n - q^n}{p - q}$  is the  $n$ -th  $(p, q)$ -integer (cf. [12, 13, 30]), it was introduced in the context of set partition statistics (cf. [33]). Note that  $\lim_{p \rightarrow 1} [n]_{p,q} = [n]_q$  and  $\lim_{p \rightarrow 1} \text{Lif}_{k,p,q}(t) = \text{Lif}_{k,q}(t)$ .

As we already mentioned the  $(p, q)$ -analogues are an extension of the  $q$ -analogues, and coincide in the limit when  $p$  tends to 1. The  $(p, q)$ -calculus was studied in [8], in connection with quantum mechanics. Properties of the  $(p, q)$ -analogues of the binomial coefficients were studied in [10]. The  $(p, q)$ -analogues of hypergeometric series, special functions, Stirling numbers, Hermite polynomials have been studied before, see for instance [13, 26, 29, 31].

The paper is divided in two parts. In Section 2 we show several combinatorial identities of the  $(p, q)$ -poly-Euler polynomials. Some of them involving the classical Euler polynomials and another special numbers and polynomials such as the Stirling numbers of the second kind, Bernoulli polynomials of order  $s$ , etc. In Section 3 we introduce the  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Cauchy polynomials of both kinds, and we generalize some well-known identities of the classical Bernoulli and Cauchy numbers and polynomials.

## 2. SOME IDENTITIES OF THE POLY-EULER POLYNOMIALS

In this section, we give several identities of the  $(p, q)$ -poly-Euler polynomials. In particular, Theorem 2 shows a relation between the  $(p, q)$ -poly-Euler polynomials and the classical Euler polynomials.

It is possible to give the first values of the  $(p, q)$ -polylogarithm function for  $k \leq 0$ . For example,

$$\begin{aligned} \text{Li}_{0,p,q}(x) &= \frac{x}{1-x}, \\ \text{Li}_{-1,p,q}(x) &= \frac{x}{(1-px)(1-qx)}, \\ \text{Li}_{-2,p,q}(x) &= \frac{x(1+pqx)}{(1-p^2x)(1-q^2x)(1-pqx)}, \\ \text{Li}_{-3,p,q}(x) &= \frac{x(p^3q^3x^2 + 2p^2qx + 2pq^2x + 1)}{(1-p^3x)(1-q^3x)(1-p^2qx)(1-pq^2x)}. \end{aligned}$$

In general, the  $(p, q)$ -polylogarithm function for  $k \leq 0$  is a rational function. Indeed, let  $k$  be a nonnegative integer then

$$\begin{aligned} \text{Li}_{-k,p,q}(x) &= \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^{-k}} = \sum_{n=1}^{\infty} [n]_{p,q}^k x^n = \sum_{n=1}^{\infty} \left( \frac{p^n - q^n}{p - q} \right)^k x^n \\ &= \frac{1}{(p - q)^k} \sum_{n=1}^{\infty} \sum_{l=0}^k \binom{k}{l} p^{nl} (-q^n)^{k-l} x^n = \frac{1}{(p - q)^k} \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \frac{p^l q^{k-l} x}{1 - p^l q^{k-l} x}. \end{aligned}$$

Note that from (9) we obtain that  $\{E_{n,p,q}^{(k)}(x)\}_{n \geq 0}$  is an Appel sequence [28]. Therefore, we have the following basic relations.

**Theorem 1.** *If  $n \geq 0$  and  $k \in \mathbb{Z}$  then*

$$\begin{aligned}
 (i) \quad E_{n,p,q}^{(k)}(x) &= \sum_{i=0}^n \binom{n}{i} E_{i,p,q}^{(k)} x^{n-i}. \\
 (ii) \quad E_{n,p,q}^{(k)}(x+y) &= \sum_{i=0}^n \binom{n}{i} E_{i,p,q}^{(k)}(x) y^{n-i}. \\
 (iii) \quad E_{n,p,q}^{(k)}(mx) &= \sum_{i=0}^n \binom{n}{i} E_{i,p,q}^{(k)}(x) (m-1)^{n-i} x^{n-i}, \quad m \geq 1. \\
 (iv) \quad E_{n,p,q}^{(k)}(x+1) - E_{n,p,q}^{(k)}(x) &= \sum_{i=0}^{n-1} \binom{n}{i} E_{i,p,q}^{(k)}(x).
 \end{aligned}$$

**Theorem 2.** *If  $n \geq 1$  we have*

$$E_{n,p,q}^{(k)}(x) = \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j E_n(x-j).$$

*Proof.* From (2) and (9) we get

$$\begin{aligned}
 \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{xt} &= \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_{p,q}^k} \cdot \frac{2e^{xt}}{1+e^t} \\
 &= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \frac{2e^{(x-j)t}}{1+e^t} \\
 &= \sum_{l=0}^{\infty} \frac{1}{[l+1]_{p,q}^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \sum_{n=0}^{\infty} E_n(x-j) \frac{t^n}{n!}.
 \end{aligned}$$

Comparing the coefficients on both sides, we get the desired result.  $\square$

**Theorem 3.** *If  $n \geq 1$  we have*

$$E_{n,p,q}^{(k)}(x) = \sum_{l=0}^{\infty} \sum_{i=0}^l \sum_{j=0}^{i+1} \frac{2(-1)^{l-i-j}}{[i+1]_{p,q}^k} \binom{i+1}{j} (l-i-j+x)^n.$$

*Proof.* By using the binomial series we get

$$\begin{aligned}
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} &= 2 \left( \sum_{l=0}^{\infty} (-1)^l e^{lt} \right) \left( \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{[l+1]_{p,q}^k} \right) e^{xt} \\
&= 2 \sum_{l=0}^{\infty} \sum_{i=0}^l \frac{(-1)^{l-i} e^{(l-i)t}}{[i+1]_{p,q}^k} (1-e^{-t})^{i+1} e^{xt} \\
&= \left( 2 \sum_{l=0}^{\infty} \sum_{i=0}^l \frac{(-1)^{l-i} e^{(l-i)t}}{[i+1]_{p,q}^k} \right) \left( \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j e^{-tj} e^{xt} \right) \\
&= 2 \sum_{l=0}^{\infty} \sum_{i=0}^l \sum_{j=0}^{i+1} \frac{(-1)^{l-i+j} e^{(l-i-j+x)t}}{[i+1]_{p,q}^k} \binom{i+1}{j} \\
&= 2 \sum_{l=0}^{\infty} \sum_{i=0}^l \sum_{j=0}^{i+1} \frac{(-1)^{l-i+j}}{[i+1]_{p,q}^k} \binom{i+1}{j} \sum_{n=0}^{\infty} (l-i-j+x)^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{i=0}^l \sum_{j=0}^{i+1} \frac{2(-1)^{l-i+j}}{[i+1]_{p,q}^k} \binom{i+1}{j} (l-i-j+x)^n \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result.  $\square$

**2.1. Some Relations with Other Special Polynomials.** Jolany et al. [14] discovered several combinatorics identities involving generalized poly-Euler polynomials in terms of Stirling numbers of the second kind  $S_2(n, k)$ , rising factorial functions  $(x)^{(m)}$ , falling factorial functions  $(x)_m$ , Bernoulli polynomials  $\mathfrak{B}_n^{(s)}(x)$  of order  $s$ , and Frobenius-Euler functions  $H_n^{(s)}(x; u)$ . We will give similar expressions in terms of  $(p, q)$ -poly-Euler polynomials. Remember that the Stirling numbers of the second kind are defined by

$$(10) \quad \frac{(e^x - 1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n, m) \frac{x^n}{n!}.$$

**Theorem 4.** *We have the following identity*

$$(11) \quad E_{n,p,q}^{(k)}(x) = \sum_{l=0}^{\infty} \sum_{i=l}^n \binom{n}{i} S_2(i, l) E_{n-i,p,q}^{(k)}(-l) (x)^{(l)}$$

where

$$(x)^{(m)} = x(x+1) \cdots (x+m-1) \quad (m \geq 1) \quad \text{with} \quad (x)^{(0)} = 1.$$

*Proof.* From (9) and (10), and by the binomial series

$$\frac{1}{(1-x)^c} = \sum_{n=0}^{\infty} \binom{c+n-1}{n} x^n$$

we get:

$$\begin{aligned}
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} &= \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}(1-(1-e^{-t}))^{-x} \\
&= \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \sum_{l=0}^{\infty} \binom{x+l-1}{l} (1-e^{-t})^l \\
&= \sum_{l=0}^{\infty} \frac{(x)^{(l)}}{l!} (1-e^{-t})^l \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \\
&= \sum_{l=0}^{\infty} (x)^{(l)} \frac{(e^t-1)^l}{l!} \left( \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} e^{-tl} \right) \\
&= \sum_{l=0}^{\infty} (x)^{(l)} \left( \sum_{n=0}^{\infty} S_2(n, l) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(-l) \frac{t^n}{n!} \right) \\
&= \sum_{l=0}^{\infty} (x)^{(l)} \sum_{n=0}^{\infty} \left( \sum_{i=0}^n \binom{n}{i} S_2(i, l) E_{n-i,p,q}^{(k)}(-l) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{\infty} \sum_{i=l}^n \binom{n}{i} S_2(i, l) E_{n-i,p,q}^{(k)}(-l) (x)^{(l)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we have (11). Note that we use the following relation

$$\binom{x+l-1}{s} = \frac{(x)^{(l)}}{s!}.$$

□

**Theorem 5.** *We have the following identity*

$$(12) \quad E_{n,p,q}^{(k)}(x) = \sum_{l=0}^{\infty} \sum_{i=l}^n \binom{n}{i} S_2(i, l) E_{n-i,p,q}^{(k)}(x)_l,$$

where

$$(x)_m = x(x-1) \cdots (x-m+1) \quad (m \geq 1) \quad \text{with} \quad (x)_0 = 1.$$

*Proof.* From (9) and (10)

$$\begin{aligned}
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} &= \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}((e^t-1)+1)^x \\
&= \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \sum_{l=0}^{\infty} \binom{x}{l} (e^t-1)^l \\
&= \sum_{l=0}^{\infty} \frac{(x)_l}{l!} (e^t-1)^l \frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \\
&= \sum_{l=0}^{\infty} (x)_l \left( \sum_{n=0}^{\infty} S_2(n,l) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right) \\
&= \sum_{l=0}^{\infty} (x)_l \sum_{n=0}^{\infty} \left( \sum_{i=0}^n \binom{n}{i} S_2(i,l) E_{n-i,p,q}^{(k)} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{\infty} \sum_{i=l}^n \binom{n}{i} S_2(i,l) E_{n-i,p,q}^{(k)} (x)_l \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we have (12). Note that we use the following relation

$$\binom{x}{s} = \frac{(x)_s}{s!}.$$

□

The Bernoulli polynomials  $\mathfrak{B}_n^{(s)}(x)$  of order  $s$  are defined by

$$(13) \quad \left( \frac{t}{e^t-1} \right)^s e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(s)}(x) \frac{t^n}{n!}.$$

It is clear that if  $s = 1$  we recover the classical Bernoulli polynomials. For some explicit formulae of these polynomials see for example [24].

**Theorem 6.** *We have the following identity*

$$(14) \quad E_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} S_2(l+s, s) \sum_{i=0}^{n-l} \frac{\binom{n-l}{i}}{\binom{l+s}{s}} \mathfrak{B}_i^{(s)}(x) E_{n-l-i,p,q}^{(k)}.$$



*Proof.* From (9) and (13)

$$\begin{aligned}
\frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} e^{xt} &= \frac{(e^t - 1)^s}{s!} \frac{t^s e^{xt}}{(e^t - 1)^s} \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right) \frac{s!}{t^s} \\
&= \left( \sum_{n=0}^{\infty} S_2(n + s, s) \frac{t^{n+s}}{(n + s)!} \right) \left( \sum_{n=0}^{\infty} \mathfrak{B}_n^{(s)}(x) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} E_{n,p,q}^{(k)} \frac{t^n}{n!} \right) \frac{s!}{t^s} \\
&= \left( \sum_{n=0}^{\infty} S_2(n + s, s) \frac{t^{n+s}}{(n + s)!} \right) \sum_{n=0}^{\infty} \left( \sum_{i=0}^n \binom{n}{i} \mathfrak{B}_i^{(s)}(x) E_{n-i,p,q}^{(k)} \right) \frac{t^n s!}{n! t^s} \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n S_2(l + s, s) \frac{t^{l+s}}{(l + s)!} \sum_{i=0}^{n-l} \binom{n-l}{i} \mathfrak{B}_i^{(s)}(x) E_{n-l-i,p,q}^{(k)} \frac{t^{n-l}}{(n-l)!} \right) \frac{s!}{t^s} \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} S_2(l + s, s) \sum_{i=0}^{n-l} \frac{\binom{n-l}{i}}{\binom{l+s}{s}} \mathfrak{B}_i^{(s)}(x) E_{n-l-i,p,q}^{(k)} \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we get (14).  $\square$

The Frobenius-Euler functions  $H_n^{(s)}(x; u)$  are defined by

$$(15) \quad \left( \frac{1 - u}{e^t - u} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x; u) \frac{t^n}{n!}.$$

**Theorem 7.** *We have the following identity*

$$(16) \quad E_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \frac{\binom{n}{l}}{(1 - u)^s} \sum_{i=0}^s \binom{s}{i} (-u)^{s-i} H_l^{(s)}(x; u) E_{n-l,p,q}^{(k)}(i).$$

*Proof.* From (9) and (15)

$$\begin{aligned}
\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t}e^{xt} &= \frac{(1-u)^s}{(e^t-u)^s}e^{xt}\frac{(e^t-u)^s}{(1-u)^s}\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \\
&= \frac{1}{(1-u)^s}\left(\sum_{n=0}^{\infty}H_n^{(s)}(x;u)\frac{t^n}{n!}\right)\sum_{i=0}^s\binom{s}{i}e^{ti}(-u)^{s-i}\frac{2\text{Li}_{k,p,q}(1-e^{-t})}{1+e^t} \\
&= \frac{1}{(1-u)^s}\left(\sum_{n=0}^{\infty}H_n^{(s)}(x;u)\frac{t^n}{n!}\right)\sum_{i=0}^s\binom{s}{i}(-u)^{s-i}\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(i)\frac{t^n}{n!} \\
&= \frac{1}{(1-u)^s}\sum_{i=0}^s\binom{s}{i}(-u)^{s-i}\left(\sum_{n=0}^{\infty}H_n^{(s)}(x;u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(i)\frac{t^n}{n!}\right) \\
&= \frac{1}{(1-u)^s}\sum_{i=0}^s\binom{s}{i}(-u)^{s-i}\sum_{n=0}^{\infty}\left(\sum_{l=0}^n\binom{n}{l}H_l^{(s)}(x;u)E_{n-l,p,q}^{(k)}(i)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^s}\sum_{l=0}^n\binom{n}{l}\sum_{i=0}^s\binom{s}{i}(-u)^{s-i}H_l^{(s)}(x;u)E_{n-l,p,q}^{(k)}(i)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we get (16).  $\square$

### 3. THE $(p, q)$ -POLY BERNOULLI POLYNOMIALS AND THE $(p, q)$ -POLY POLY-CAUCHY POLYNOMIALS

In this section we introduce the  $(p, q)$ -poly Bernoulli polynomials by means of the  $(p, q)$ -polylogarithm function and the  $(p, q)$ -poly Cauchy polynomials by using the  $(p, q)$ -integral. In general it is not difficult to extend the results of [20].

The  $(p, q)$ -derivative of the function  $f$  is defined by (cf. [4, 12])

$$D_{p,q}f(x) = \begin{cases} \frac{f(px)-f(qx)}{(p-q)x}, & \text{if } x \neq 0, \\ f'(0), & \text{if } x = 0. \end{cases}$$

In particular if  $p \rightarrow 1$  we obtain the  $q$ -derivative [1]. The  $(p, q)$ -integral of the function  $f$  is defined by

$$\int_0^x f(t)d_{p,q}t = \begin{cases} (q-p)x \sum_{n=0}^{\infty} \frac{p^n}{q^{n+1}} f\left(\frac{p^n}{q^{n+1}}x\right), & \text{if } |p/q| < 1; \\ (p-q)x \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x\right), & \text{if } |p/q| > 1. \end{cases}$$

For example,

$$\int_0^1 t^l d_{p,q}t = \frac{1}{[l+1]_{p,q}}.$$

We introduce the  $(p, q)$ -poly Bernoulli polynomials by

$$\frac{\text{Li}_{k,p,q}(1 - e^{-t})}{1 - e^{-t}} e^{-xt} = \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}).$$

In particular,  $\lim_{p \rightarrow 1} B_{n,p,q}^{(k)}(x) = B_{n,q}^{(k)}(x)$ , which are the  $q$ -poly-Bernoulli polynomials studied recently in [20].

The following theorem related the  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Euler polynomials.

**Theorem 8.** *If  $n \geq 1$  we have*

$$E_{n,p,q}^{(k)}(x) + E_{n,p,q}^{(k)}(x+1) = 2B_{n,p,q}^{(k)}(-x) - 2B_{n,p,q}^{(k)}(1-x).$$

*Proof.* From the following equality

$$\frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 + e^t} (1 + e^t) e^{xt} = \frac{2\text{Li}_{k,p,q}(1 - e^{-t})}{1 - e^{-t}} (1 - e^{-t}) e^{xt}$$

we obtain

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} E_{n,p,q}^{(k)}(x+1) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(-x) \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(1-x) \frac{t^n}{n!}.$$

Comparing the coefficients on both sides, we get the desired result.  $\square$

The weighted Stirling numbers of the second kind,  $S_2(n, m, x)$ , were defined by Carlitz [5] as follows

$$\frac{e^{xt}(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n, m, x) \frac{t^n}{n!}.$$

**Theorem 9.** *If  $n \geq 1$ , we have*

$$B_{n,p,q}^{(k)}(x) = \sum_{m=0}^n \frac{(-1)^{m+n} m!}{[m+1]_{p,q}^k} S_2(n, m, x).$$

*Proof.*

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x) \frac{t^n}{n!} &= \frac{\text{Li}_{p,q}(1 - e^{-t})}{1 - e^{-t}} e^{-xt} \\
&= \sum_{m=0}^{\infty} \frac{(1 - e^{-t})^m}{[m+1]_{p,q}^k} e^{-xt} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m m!}{[m+1]_{p,q}^k} \cdot \frac{(e^{-t} - 1)^m}{m!} e^{-xt} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m m!}{[m+1]_{p,q}^k} \cdot \sum_{n=m}^{\infty} S_2(n, m, x) \frac{(-t)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-1)^{m+n} m!}{[m+1]_{p,q}^k} S_2(n, m, x) \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we get the desired result.  $\square$

The  $(p, q)$ -poly-Cauchy polynomials of the first kind are defined by

$$(17) \quad C_{n,p,q}^{(k)}(x) = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (t_1 \cdots t_k - x)_n d_{p,q} t_1 \cdots d_{p,q} t_k.$$

Note that  $\lim_{p \rightarrow 1} C_{n,p,q}^{(k)}(x) = C_{n,q}^{(k)}(x)$ , i.e., we obtain the  $q$ -poly-Cauchy polynomials [20, 17].

Remember that the (unsigned) Stirling numbers of the first kind are defined by

$$(18) \quad \frac{(\ln(1+x))^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} S_1(n, m) \frac{x^n}{n!}.$$

Moreover, they satisfy (cf. [9])

$$(19) \quad x^{(n)} = x(x+1) \cdots (x+n-1) = \sum_{m=0}^n S_1(n, m) x^m.$$

The weighted Stirling numbers of the first kind,  $S_1(n, m, x)$ , are defined by ([5])

$$\frac{(1-t)^{-x} (-\ln(1-t))^m}{m!} = \sum_{n=m}^{\infty} S_1(n, m, x) \frac{t^n}{n!}.$$

**Theorem 10.** *If  $n \geq 1$ , we have*

$$(20) \quad C_{n,p,q}^{(k)}(x) = \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k}$$

$$(21) \quad = \sum_{m=0}^n S_1(n, m, x) \frac{(-1)^{n-m}}{[m+1]_{p,q}^k}.$$

*Proof.* By (17), (19) and  $(x)_n = (-1)^n (-x)^{(n)}$ , we have

$$\begin{aligned} C_{n,p,q}^{(k)}(x) &= \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \underbrace{\int_0^1 \cdots \int_0^1}_{k} (t_1 \cdots t_k - x)^m d_{p,q} t_1 \cdots d_{p,q} t_k \\ &= \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} (-x)^{m-l} \underbrace{\int_0^1 \cdots \int_0^1}_{k} t_1^l \cdots t_k^l d_{p,q} t_1 \cdots d_{p,q} t_k \\ &= \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^{m-l}}{[l+1]_{p,q}^k} \\ &= \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k}. \end{aligned}$$

Comparing the coefficients on both sides, we get (20). Finally, from the following relation ([5, Eq. (5.2)])

$$S_1(n, m, x) = \sum_{i=0}^n \binom{m+i}{i} x^i S_1(n, m+i),$$

we have

$$\begin{aligned}
C_{n,p,q}^{(k)}(x) &= \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \\
&= \sum_{l=0}^n \sum_{m=l}^n (-1)^{n-m} S_1(n, m) \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \\
&= \sum_{l=0}^n \sum_{m=l}^{n+l} (-1)^{n-m} S_1(n, m) \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \\
&= \sum_{l=0}^n \sum_{m=0}^n (-1)^{n-m+l} S_1(n, m+l) \binom{m+l}{l} \frac{(-x)^l}{[m+1]_{p,q}^k} \\
&= \sum_{m=0}^n \frac{(-1)^{n-m}}{[m+1]_{p,q}^k} \sum_{l=0}^m \binom{m+l}{l} S_1(n, m+l) x^l \\
&= \sum_{m=0}^n \frac{(-1)^{n-m}}{[m+1]_{p,q}^k} S_1(n, m, x).
\end{aligned}$$

□

It is not difficult to give a  $(p, q)$ -analogue of (8).

**Theorem 11.** *The exponential generating function of the  $(p, q)$ -poly-Cauchy polynomials  $C_{n,p,q}^{(k)}(x)$  is*

$$(22) \quad \frac{\text{Lif}_{k,p,q}(\ln(1+t))}{(1+t)^x} = \sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!},$$

where

$$(23) \quad \text{Lif}_{k,p,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! [n+1]_{p,q}^k}$$

is the  $k$ -th  $(p, q)$ -polylogarithm factorial function.

*Proof.* From Theorem 10 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{n,p,q}^{(k)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^{n-m} S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (-1)^{n-m} S_1(n, m) \frac{t^n}{n!} \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \\
&= \sum_{m=0}^{\infty} \frac{(\ln(1+t))^m}{m!} \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k} \\
&= \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} \sum_{m=l}^{\infty} \frac{(\ln(1+t))^m}{(m-l)! [m-l+1]_{p,q}^k} \\
&= \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} \sum_{n=0}^{\infty} \frac{(\ln(1+t))^{n+l}}{n! [n+1]_{p,q}^k} \\
&= \frac{1}{(1+t)^x} \sum_{n=0}^{\infty} \frac{(\ln(1+t))^n}{n! [n+1]_{p,q}^k} \\
&= \frac{\text{Lif}_{k,p,q}(\ln(1+t))}{(1+t)^x}.
\end{aligned}$$

□

Similarly, we can defined the  $(p, q)$ -poly-Cauchy polynomials of the second kind by

$$\widehat{C}_{n,p,q}^{(k)}(x) = \underbrace{\int_0^1 \cdots \int_0^1}_k (-t_1 \cdots t_k + x)_n d_{p,q} t_1 \cdots d_{p,q} t_k.$$

We can find analogous expressions to (20), (21) and (22).

**Theorem 12.** *If  $n \geq 1$ , we have*

$$(24) \quad \widehat{C}_{n,p,q}^{(k)}(x) = (-1)^n \sum_{m=0}^n S_1(n, m) \sum_{l=0}^m \binom{m}{l} \frac{(-x)^l}{[m-l+1]_{p,q}^k}$$

$$(25) \quad = (-1)^n \sum_{m=0}^n S_1(n, m, -x) \frac{1}{[m+1]_{p,q}^k}.$$

Moreover, the exponential generating function of the  $(p, q)$ -poly-Cauchy polynomials  $\widehat{C}_{n,p,q}^{(k)}(x)$  is

$$(1+t)^x \text{Lif}_{k,p,q}(-\ln(1+t)) = \sum_{n=0}^{\infty} \widehat{C}_{n,p,q}^{(k)}(x) \frac{t^n}{n!}.$$

**3.1. Some relations between  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Cauchy polynomials.** The weighted Stirling numbers satisfy the following orthogonality relation [5]:

$$\sum_{l=m}^n (-1)^{n-l} S_2(n, l, x) S_1(l, m, x) = \sum_{l=m}^n (-1)^{l-m} S_1(n, l, x) S_2(l, m, x) = \delta_{m,n},$$

where  $\delta_{m,n} = 1$  if  $m = n$  and 0 otherwise. From above relations we obtain the inverse relation:

$$f_n = \sum_{m=0}^n (-1)^{n-m} S_1(n, m, x) g_m \iff g_n = \sum_{m=0}^n S_2(n, m, x) f_m.$$

**Theorem 13.** *The  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Cauchy polynomials of both kinds satisfy the following relations*

$$(26) \quad \sum_{m=0}^n S_1(n, m, x) B_{m,p,q}^{(k)}(x) = \frac{n!}{[n+1]_{p,q}^k},$$

$$(27) \quad \sum_{m=0}^n S_2(n, m, x) C_{m,p,q}^{(k)}(x) = \frac{1}{[n+1]_{p,q}^k},$$

$$(28) \quad \sum_{m=0}^n S_2(n, m, -x) \widehat{C}_{m,p,q}^{(k)}(x) = \frac{(-1)^n}{[n+1]_{p,q}^k}.$$

*Proof.* From Theorem 9 and the inverse relation for the weighted Stirling numbers with

$$f_m = \frac{(-1)^m m!}{[m+1]_{p,q}^k}, \quad \text{and} \quad g_n = (-1)^n B_{n,p,q}^{(k)}(x),$$

we obtain the identity (26). The remaining relations can be verified in a similar way by using Theorems 10 and 12.  $\square$

Note that if  $p \rightarrow 1$  we obtain Theorem 6 in [20].

**Theorem 14.** *The  $(p, q)$ -poly-Bernoulli polynomials and  $(p, q)$ -poly-Cauchy polynomials of both kinds satisfy the following relations*

$$(29) \quad B_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^n (-1)^{n-m} m! S_2(n, m, x) S_2(m, l, y) C_{l,p,q}^{(k)}(y),$$

$$(30) \quad B_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^n (-1)^n m! S_2(n, m, x) S_2(m, l, -y) \widehat{C}_{l,p,q}^{(k)}(y),$$

$$(31) \quad C_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^n \frac{(-1)^{n-m}}{m!} S_1(n, m, x) S_1(m, l, y) B_{l,p,q}^{(k)}(y),$$

$$(32) \quad \widehat{C}_{n,p,q}^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^n \frac{(-1)^n}{m!} S_1(n, m, -x) S_1(m, l, y) B_{l,p,q}^{(k)}(y).$$



*Proof.* We only show the proof of (31). The proofs of the remaining identities are similar. From Equations (21) and (26) we have

$$\begin{aligned}
\sum_{l=0}^n \sum_{m=0}^n \frac{(-1)^{n-m}}{m!} S_1(n, m, x) S_1(m, l, y) B_{l,p,q}^{(k)}(y) \\
= \sum_{m=0}^n \frac{(-1)^{n-m}}{m!} S_1(n, m, x) \sum_{l=0}^m S_1(m, l, y) B_{l,p,q}^{(k)}(y) \\
= \sum_{m=0}^n \frac{(-1)^{n-m}}{m!} S_1(n, m, x) \frac{m!}{[m+1]_{p,q}^k} \\
= C_{n,p,q}^{(k)}(x). \quad \square
\end{aligned}$$

Finally, we show some relations between  $(p, q)$ -poly-Cauchy polynomials of both kinds.

**Theorem 15.** *If  $n \geq 1$  we have*

$$(33) \quad (-1)^n \frac{C_{n,p,q}^{(k)}(x)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{C}_{m,p,q}^{(k)}(x)}{m!},$$

$$(34) \quad (-1)^n \frac{\widehat{C}_{n,p,q}^{(k)}(x)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{C_{m,p,q}^{(k)}(x)}{m!}.$$

*Proof.* From definition of the  $(p, q)$ -poly-Cauchy polynomials of the first kind we get

$$\begin{aligned}
(-1)^n \frac{C_{n,p,q}^{(k)}(x)}{n!} &= (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_k \frac{(t_1 \cdots t_k - x)_n}{n!} d_{p,q} t_1 \cdots d_{p,q} t_k \\
&= (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_k \binom{t_1 \cdots t_k - x}{n} d_{p,q} t_1 \cdots d_{p,q} t_k \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_k \binom{x - t_1 \cdots t_k + n - 1}{n} d_{p,q} t_1 \cdots d_{p,q} t_k
\end{aligned}$$

By using the Vandermonde convolution

$$\sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$$

with  $r = x - t_1 \cdots t_k$  and  $s = n - 1$  we obtain

$$\begin{aligned}
(-1)^n \frac{C_{n,p,q}^{(k)}(x)}{n!} &= \underbrace{\int_0^1 \cdots \int_0^1}_k \sum_{l=0}^n \binom{x - t_1 \cdots t_k}{l} \binom{n-1}{n-l} d_{p,q} t_1 \cdots d_{p,q} t_k \\
&= \sum_{l=0}^n \binom{n-1}{n-l} \underbrace{\int_0^1 \cdots \int_0^1}_k \binom{x - t_1 \cdots t_k}{l} d_{p,q} t_1 \cdots d_{p,q} t_k \\
&= \sum_{l=0}^n \binom{n-1}{n-l} \frac{1}{l!} \underbrace{\int_0^1 \cdots \int_0^1}_k (-t_1 \cdots t_k + x)_l d_{p,q} t_1 \cdots d_{p,q} t_k \\
&= \sum_{l=0}^n \binom{n-1}{n-l} \frac{\widehat{C}_{l,p,q}^{(k)}(x)}{l!}.
\end{aligned}$$

The proof of (34) is similar. □

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